THE ANALYST.

VOL. V. NOVEMBER, 1878. No. 6.

SOLUTION OF THE EQUATION OF THE FIFTH DEGREE.

TRANSLATED FROM THE THEORY OF ELLIPTIC FUNCTIONS OF BRIOT AND BOUQUET, SECOND EDITION.

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421. IT was announced by Galois that, so far as $n = 11$, the degree of the modular equation may be lowered by unity. M M. Hermite and Betti have given two demonstrations of this theorem based upon different principles. Let us consider a symmetric or alternate function (v_a, v_b) of two roots of the modular equation, then, the same function of two other roots, and so on as far as the last two roots, and let us designate by U a symmetrical function of these $\frac{1}{2}(n+1)$ quantities, we easily demonstrate, by the aid of the laws of permutation before established, that when n does not surpass 11, among the different manners of associating the roots, two by two, there obtain those for which the function U only acquires n values for each value of u and by consequence satisfies an equation of the degree n.

About each one of the critical points a root v_a remains holomorphic and the associated root v_b acquires the n other values; the quantity (v_a, v_b) and in consequence the function U takes n values, forming a circular system.

If this function only acquires n values in the whole extent of the plan, it is impossible that two of these values can have a common element (v_a, v_b); for by a suitable way, v_a becomes equal to the root which remains holomorphic about one of the critical points; if two values of U had a common element (v_a, v_b) without being identical, they would engender two circular systems of n values each.

We may associate two roots v_a and v_b taken at pleasure; for, about the critical point where v_a remains holomorphic, v_b acquires the n other values.

In a single value of the function U two elements cannot present the same difference of indices, because if we should describe the contour or lacing ($\bigcirc$) [*le lacet*] a certain number of times one of the elements would become equal to the other. Since we have perceived that the contour (a_0) causes the function U to acquire the same values as the loop ($\bigcirc$), we are certain that this function has only n values in the whole plan, the other lacings releading to the loops (a_0) and ($\bigcirc$).

422. These considerations allow us to find easily the favorable ways.

We should associate the roots by way of difference and form the product of the $\frac{1}{2}(n+1)$ quantites.

For $n = 5$ if we take as first factor $V—v_0$, the combination

$$U = (V-v_0)(v_1-v_4)(v_2-v_3) \tag{1}$$

is the only one in which the difference of the indices in the last factors is not the same.

By the contour ($\bigcirc$) this function acquires the five values

$$\left\{\begin{aligned} U_0 &= (V-v_0)(v_1-v_4)(v_2-v_3),\\ U_1 &= (V-v_1)(v_2-v_0)(v_3-v_4),\\ U_2 &= (V-v_2)(v_3-v_1)(v_4-v_0),\\ U_3 &= (V-v_3)(v_4-v_2)(v_0-v_1),\\ U_4 &= (V-v_4)(v_0-v_3)(v_1-v_2). \end{aligned}\right. \tag{2}$$

After the law of permutation written in number 408 the contour (a_0) reproduces the same values in another order; from that we conclude that the function U has only the five preceding values, and by consequence that it satisfies an algebraic equation between u and U of the fifth degree in U.

For $n = 7$ there are two favorable combinations,

$$U = (V-v_0)(v_2-v_3)(v_4-v_6)(v_1-v_5), \tag{3}$$

$$U = (V-v_0)(v_2-v_6)(v_5-v_4)(v_1-v_3). \tag{4}$$

We obtain them in the following manner; upon the contour (a_0) the law of permutation is $(V, v_5, v_6, v_4, v_3, v_1, v_2)$.

Let us take as first factor $V—v_0$, and as the second $v_2—v_a$, for example $v_2—v_3$; when we describe the noose (a_0) the product $(V—v_0)(v_2—v_3)$ becomes $(v_5—v_0)(V—v_1)$: the first product being completed, before reproducing the second, the contour ($\bigcirc$) once run over, will contain the factor $v_4—v_6$, which gives origin to $v_5—v_0$, and by consequence the last factor will be $v_1—v_5$. By the noose ($\bigcirc$) the function (3) acquires seven values: these same values reproduce themselves upon the loop (a_0); whence we conclude that the function U satisfies an algebraic equation between u and U, of the seventh degree in U. The function (4) posseses the same property.

For $n = 11$, we have also two favorable combinations

(5) $U = (V - v_0)(v_{10} - v_5)(v_3 - v_6)(v_9 - v_7)(v_2 - v_1)(v_4 - v_8),$

(6) $U = (V - v_0)(v_{10} - v_9)(v_8 - v_5)(v_7 - v_3)(v_6 - v_1)(v_2 - v_4),$

which we obtain in the same manner.

Upon the contour (a_0) the law of permutations is $(V, v_1, v_6, v_4, v_3, v_9, v_2, v_8, v_7, v_5, v_{10})$.

We will take as first factor $V - v_0$, and as second $v_{10} - v_a$. For $n = 13$ there is no favorable combination.

423. From the depression of the degree of the modular equation for $n = 5$, M. Hermite has deduced a method for the resolution of the equation of the fifth degree by elliptic functions (*Comptes rendus*, t. XVIII).

Let us form the equation of the fifth degree in U. The values of U being finite for all finite values of u, the coefficient of U^5 is equal to unity. For $u = \infty$ the six values of v are infinite, one of the degree 5, the others of the degree $\frac{1}{5}$ (number 408): the five values of U are infinite and of the degree $\frac{27}{5}$; the sum of the negative orders of the function U being equal to 27, the equation is of the 27th degree in relation to u (number 135).

For values of u exceedingly small the approximate values of v are

$$V = -2^{-2}u^5,\ v_0 = 2^{\frac{2}{5}}u^{\frac{1}{5}},\ v_1 = v_0 e^{\frac{2\pi i}{5}},\ v_2 = v_0 e^{2\frac{2\pi i}{5}},\ v_3 = v_0 e^{3\frac{2\pi i}{5}},$$
$$v_4 = v_0 e^{4\frac{2\pi i}{5}},$$

and consequently those of U are,

(7) $$U_0 = 2^{\frac{6}{5}}5^{\frac{1}{2}}u^{\frac{3}{5}},\ U_1 = U_0 e^{\frac{6\pi i}{5}},\ U_2 = U_0 e^{2\frac{6\pi i}{5}},\ U_3 = U_0 e^{3\frac{6\pi i}{5}},$$
$$U_4 = U_0 e^{4\frac{6\pi i}{5}}.$$

We have put $\xi = \frac{v}{u^5}$; putting in the same way $\Phi = \frac{U}{u^{15}} = (\xi - \xi_0) \times (\xi_1 - \xi_4)(\xi_2 - \xi_3)$, the coefficients of the equation in ξ being entire polynomes u^8 (number 401), those of the equation in Φ will possess the same property; the values of Φ only becoming infinite for $u = 0$, this equation is of the form

$$u^{8\beta_0}\Phi^5 + \sum_{p=1}^{p=4} u^{8\beta_p}(a_p + b_p u^8 + c_p u^{16} + \ldots)\Phi^{5-p} + (a_5 + b_5 u^8 + c_5 u^{16} + \ldots) = 0.$$

Replacing Φ by $U \div u^{15}$ and multiplying by u^3, we obtain the equation

$$u^{8(\beta_0 - 9)}U^5 + \sum_{p=1}^{p=4} u^{8(\beta_p - 9) + 15p}(a_p + b_p u^8 + \ldots)U^{5-p} + u^3(a_5 + a_5 u^8 + \ldots) = 0.$$

For very small values of u, the values of U being very small of the degree $\frac{3}{5}$, and forming a circular system, we have $\beta_0 = 9$, $a_5 = -2^6 5^{\frac{5}{2}}$, $8(\beta_p - 9) + 15p > 3$, and in consequence $\beta_p =$ or $> 10 - 2p$; we will take $\beta_p = 10 - 2p$. The equation sought for is then of the form

$$(8)\quad U^5+\sum_{p=1}^{p=4} u^{8-p}(a_p+b_pu^8+c_pu^{16})U^{5-p}+u^3(a_5+b_5u^8+c_5u^{16}+d_5u^{24})=0.$$

If we consider the function $(U)=[(V)-(v_0)]\,[(v_1)-(v_4)]\,[(v_2)-(v_3)]$, relative to the variable $u_1=1\div u$, we have

$$(U)=\frac{-1}{Vv_0v_1v_2v_3v_4}(V-v_2)(v_3-v_1)(v_4-v_0)=\frac{U}{u^6},$$

since the product of the roots of the modular equation (number 411) is equal to $-u^6$.

So the equation (8) ought not to change when we replace in it u by $1\div u$, and U by $U\div u^6$. But, by this substitution the equation becomes

$$U^5+\sum_{p=1}^{p=4} u^{7p-8}(a_p+b_pu^{-8}+c_pu^{-16})U^{5-p}+u^3(d_5+c_5u^8+b_5u^{16}+a_5u^{24})=0,$$

and we hence conclude that it contains no term in U^4, and that we have $b_2=c_2=0$, $c_3=0$, $b_3=a_3$, $c_4=a_4$, $d_5=a_5$, $c_5=b_5$. The equation reduces to

$$(9)\quad \left\{\begin{aligned} &U^5+a_2u^6U^3+a_3u^5(1+u^8)U^2+u^4(a_4+b_4u^8+a_4u^{16})U\\ &\qquad\qquad +u^5(a_5+b_5u^8+b_5u^{16}+a_5u^{24})=0.\end{aligned}\right.$$

For $u=1$, the root v_0 of the modular equation is equal to $+1$, and the five others to -1 (number 406); the five values of U reduce themselves to zero; the polynomes in u, coefficients of the diverse powers of U in the equation (9) ought to become null for $u=1$; and we deduce $a_2=0$, $a_3=0$, $b_4=-2a_4$, $b_5=-a_5$, which reduces the equation to the simple form

$$(10)\qquad U^5+a_4u^4(1-u^8)^2U+a_5u^3(1-u^8)^2(1+u^8)=0.$$

We know the coefficient a_5, it remains to determine the coefficient a_4; we follow the road which has been pointed out in number 411 for the calculation of the modular equation.

To a real, positive, and very small value of u, corresponds a value of ρ of the form $\rho=si$, s being positive and very great, and in consequence a value of q real, positive, and very small.

By developing $\varphi(\rho)$ in series and restricting oneself to the first two terms there results

$$\begin{aligned}
u&=2^{\frac12}q^{\frac18}(1-q),\quad v_0=2^{\frac12}q^{\frac1{40}}(1-q^{\frac15}),\\
v_1&=2^{\frac12}q^{\frac1{40}}\left(e^{\frac{2\pi i}{5}}-q^{\frac15}e^{-\frac{2\pi i}{5}}\right),\\
v_2&=2^{\frac12}q^{\frac1{40}}\left(e^{\frac{4\pi i}{5}}-q^{\frac15}e^{-\frac{4\pi i}{5}}\right),\\
v_3&=2^{\frac12}q^{\frac1{40}}\left(e^{-\frac{4\pi i}{5}}-q^{\frac15}e^{\frac{4\pi i}{5}}\right),\\
v_4&=2^{\frac12}q^{\frac1{40}}\left(e^{-\frac{2\pi i}{5}}-q^{\frac15}e^{\frac{2\pi i}{5}}\right),
\end{aligned}$$

(5) $\quad U=(V-v_0)(v_{10}-v_5)(v_3-v_6)(v_9-v_7)(v_2-v_1)(v_4-v_8),$

(6) $\quad U=(V-v_0)(v_{10}-v_9)(v_8-v_5)(v_7-v_3)(v_6-v_1)(v_2-v_4),$

which we obtain in the same manner.

Upon the contour (a_0) the law of permutations is $(V, v_1, v_6, v_4, v_3, v_9, v_2, v_8, v_7, v_5, v_{10})$.

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For very small values of u, the values of U being very small of the degree $\frac{3}{5}$, and forming a circular system, we have $\beta_0=9$, $a_5=-2^6 5^{\frac{5}{2}}$, $8(\beta_p-9)+15p>3$, and in consequence $\beta_p=$ or $>10-2p$; we will take $\beta_p=10-2p$. The equation sought for is then of the form

$$u = 2^{\frac{1}{2}} q^{\frac{1}{8}}(1-q+2q^2-3q^3+4q^4),$$

$$v_0 = 2^{\frac{1}{2}} q^{\frac{4}{56}}(1-q^{\frac{1}{7}}+2q^{\frac{2}{7}}-3q^{\frac{3}{7}}+4q^{\frac{4}{7}}).$$

We find also for the first combination

$$U = i2^2 7^{\frac{1}{2}} q^{\frac{1}{14}}\left(1-\frac{1-i\sqrt{7}}{2} q^{\frac{2}{7}}+q^{\frac{4}{7}}\right),$$

whence

$$a_7 = i2^{12}7^{\frac{7}{2}},\ a_5 = 0,\ a_3 = -i2^4 7^{\frac{5}{2}}\frac{1-i\sqrt{7}}{2},\ a_6 = 2^8 7^4 \left[1-\left(\frac{1-i\sqrt{7}}{1}\right)\right].$$

We deduce the second combination from the first in replacing i by $-i$.

The following is extracted from Todhunter's Theory of Equations, sections 340, 341.

"The general equation of the fifth degree can always be reduced to one of the following forms:

"$x^5+px+q=0$; $x^5+px^2+q=0$; $x^5+px^3+q=0$; $x^5+px^4+q=0$. [Due to Mr. Jerrard or to E. S. Bring. Quarterly Journal of Mathematics, Vol. VI.]

"Mr. Jerrard considered that the algebraical solution of equations of the 5th degree could be effected; his proposed method formed the subject of an enquiry by Sir W. R. Hamilton in the *Reports of the British Association*, Vol. VI. Most mathematicians admit that Abel has demonstrated the impossibility of the algebraical solution of equations of a higher degree than the fourth. An abstract of Sir W. R. Hamilton's exposition of *Abel's* argument will be found in the *Quarterly Journal of Mathematics*, Vol. V. A simpler demonstration due to *Wantzel* will be found in *Serret's Cours d' Algebre Supérieure*." [See Analyst, p. 65–70, Vol. IV.] "An *Essay* on the resolution of algebraical equations by the late *Judge Hargreave* has been recently published: the results arrived at are to some extent at variance with those of *Abel* and *Sir W. R. Hamilton*."

TO DRAW A CIRCLE TANGENT TO THREE GIVEN CIRCLES.

BY ISAAC H. TURRELL, CINCINNATI, OHIO.

This problem is a celebrated one in the history of pure geometry. When or by whom it was first proposed I am unable to ascertain; but the best mathematicians of modern times have not thought it unworthy